SOME SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL MAGNETOHYDRODYNAMICS AND THEIR APPLICATION TO PROBLEMS OF SHOCK WAVE PROPAGATION

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This paper points out cases of integrability of the equations describing one-dimensional motion of an electrically conducting gas with cylindrical and plane symmetry, the case of cylindrical symmetry being considered in greater detail. For steady motions with infinite conductivity a general solution of the equations is found, and a short description is given of the corresponding flows.

Unsteady self-similar and non-self-similar motions associated with shock waves are considered. A method is given for joining the solutions [1-3] to gas at rest by means of a shock wave. Concrete cases are solved which may have application to the theory of impulsive gaseous discharge.

1. We will consider the motion to be one-dimensional with cylindrical or plane symmetry. All functions characterizing the motion will depend on the one geometric coordinate r and time t.

For the case of a perfect gas of finite conductivity with viscosity and heat conductivity neglected, we have for the unknown quantities the system

$$-\rho\frac{dv}{dt}=\frac{\partial p^*}{\partial r}+\frac{2(\nu-1)h_\varphi}{r},\qquad -\frac{1}{\rho}\frac{d\rho}{dt}=\frac{\partial v}{\partial r}+\frac{(\nu-1)v}{r}$$

$$-\frac{1}{2}\frac{dh_z}{dt}=h_z\left(\frac{\partial v}{\partial r}+\frac{(\nu-1)v}{r}\right)-r^{1-\nu}h_z^{1/2}\frac{\partial}{\partial r}\left(\nu_m r^{\nu-1}\frac{\partial h_z^{1/2}}{\partial r}\right) \qquad (1.1)$$

$$-\frac{1}{2}\frac{dh_\varphi}{dt}=h_\varphi\frac{\partial v}{\partial r}-h_\varphi^{1/2}\frac{\partial}{\partial r}\left[\nu_m r^{-1}\frac{\partial}{\partial r}(rh_\varphi^{1/2})\right]$$

$$-\frac{dp}{dt}=\gamma p\left(\frac{\partial v}{\partial r}+\frac{(\nu-1)v}{r}\right)-2(\gamma-1)\nu_m\left\{\frac{1}{r^2}\left[\frac{\partial}{\partial r}(rh_\varphi^{1/2})\right]^2+\left(\frac{\partial h_z^{1/2}}{\partial r}\right)^2\right\} \qquad (1.2)$$

$$\left(p^*=p+h,\quad h=h_z+(\nu-1)h_\varphi,\quad h_z=\frac{H_z^2}{8\pi},\quad h_\varphi=\frac{H_\varphi^2}{8\pi}\right)$$

Here H_z and H_ϕ are the components of the magnetic field intensity vector, ν_m is the magnetic viscosity, $\nu = 2$ for the case of cylindrical symmetry and $\nu = 1$ for motion with plane waves; the remaining symbols are conventional or are obvious from the equations. The magnetic field intensity vector **H** is always perpendicular to the velocity vector. For $\nu = 1$ the azimuthal component of the field should vanish ($h_\phi = 0$). In place of one of the equations of the system (1.1)-(1.2), for example, the first of Equations (1.1), one can take the equation expressing the law of conservation of energy [4]

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}+h\right)+r^{1-\nu}\frac{\partial}{\partial r}\left\{r^{\nu-1}\left[v\left(\frac{\rho v^2}{2}+\frac{\gamma p}{\gamma-1}+2h\right)-\right.\right.$$
$$\left.\left.-\nu_m\left(2(\nu-1)\frac{h_\varphi^{1/2}}{r}\frac{\partial}{\partial r}(rh_\varphi^{1/2})+\frac{\partial h_z}{\partial r}\right)\right]\right\}=0 \qquad (1.3)$$

In the case of infinite conductivity the system of equations (1.1)-(1.2) simplifies because $\nu_m = 0$.

For homothermal flows, that is, flows with zero temperature gradient [1], Equation (1.2) is replaced by the equation

$$\partial T/\partial r = 0 \quad \text{or} \quad p = \theta(t)\rho \qquad (1.4)$$

For unsteady motions of an ideal medium including shock waves, the conditions of conservation of mass, momentum, continuity of the electrical field, and energy must be satisfied. For propagation of a shock wave into a quiescent medium they have the form

$$\rho_2(v_2-u) = -\rho_1 u, \qquad v_2\rho_2(v_2-u)+p_2^* = p_1^* \qquad (u = dr_2/dt) \qquad (1.5)$$

$$h_{z2}\rho_1^2 = h_{z1}\rho_2^2, \qquad h_{\varphi 2}\rho_1^2 = h_{\varphi 1}\rho_2^2 \qquad (1.6)$$

$$(v_2-u)\left(\frac{\rho_2 v_2^2}{2}+\frac{p_2}{\gamma-1}+h_2\right)+v_2 p_2^* = -u\left(\frac{p_1}{\gamma-1}+h_1\right) \qquad (1.7)$$

where the subscript 1 denotes quantities in the undisturbed medium and the subscript 2 quantities behind the shock wave front, u is the speed of the wave, and $r_2(t)$ its radius.

2. In the steady case the system (1.1)-(1.3) with $\nu_m = 0$, $\nu = 2$ can be integrated completely, having the five integrals *

$$p = c_1\rho^\gamma, \quad \rho v r = c_2, \quad h_\varphi = c_3 r^2\rho^2, \quad h_z = c_4\rho^2, \quad \frac{v^2}{2}+\frac{\gamma p}{(\gamma-1)\rho}+\frac{2h}{\rho} = c_5 \qquad (2.1)$$

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* The possibility of integrating the system (1.1)-(1.2) in the stationary case when there is only one of the components h_z or h_ϕ was pointed out by K.P. Staniukovich [5].

where $c_1, \ldots, c_5$ are arbitrary constants.

A characteristic singularity of the flows under consideration is the presence in the stream of two limit lines on which

$$\frac{\partial v}{\partial r} = \infty, \qquad v = a^* = \left(\frac{\gamma p + 2h}{\rho}\right)^{1/2}$$

that is, the speed of the gas is equal to the total speed of sound*. If r_0 and $r_1 (r_0 < r_1)$ are the radii of the limit circles, then in the general case flow is possible only in the region enclosed between the cylinders with radii r_0 and r_1, where either $v < a^*$ or $v > a^*$.

The dependence of the radii r_0, r_1 and of the value of the density at the limit lines on the constants $c_1, \ldots, c_5$ is given by the relations

$$(\gamma c_1 \rho^{\gamma-2} + 2c_4 + 2c_3 r^2)\rho^3 r^2 = c_2^2, \qquad \frac{\gamma(\gamma+1)}{2(\gamma-1)} c_1 \rho^{\gamma-1} + 3\rho(c_4 + c_3 r^2) = c_5$$

Investigation of the solution (2.1) shows that in the subsonic (supersonic) regime as r increases from r_0 to r_1 the speed may at first decrease (increase)to a certain minimum (maximum), and then increase (decrease) to the value a^*.

In the more general case of finite conductivity the integration in closed form cannot be carried out.

We note that in the case under consideration two finite algebraic integrals can be found for the system (1.1)-(1.2)

$$\rho v r^{\nu-1} = M_1, \qquad \frac{v^2}{2} + \frac{\gamma p}{(\gamma-1)\rho} + (\nu-1) M_2 r^{\nu-1} h_\varphi^{1/2} + M_3 h_z^{1/2} = M_4$$

where $M_1, \ldots, M_4$ are arbitrary constants. In the case of isothermal steady flows with infinite conductivity for $\nu = 2$ the problem of integrating the system of equations (1.1) and (1.4) leads to the solution of one first-order ordinary differential equation, which is easily integrated when $h_\phi = 0$.

In the case $\gamma = 2$, $\nu_m = 0$ the solution of the system of equations (1.1) and (1.2) also simplifies, since it has the integral

$$p = \Phi_1(\xi) h_z$$

where $\Phi_1(\xi)$ is an arbitrary function of the Lagrangean coordinate ξ. It is easy to show that if $h_\Phi = 0$ then any solution of the equations of ordinary gas dynamics permits one to construct a solution of the system (1.1) and (1.2), having, in addition, one arbitrary function. For this

* This fact was also noted in [5].

it suffices to take

$$v = v_0, \quad \rho = \rho_0, \quad p^* = p_0, \quad p = p^* - h_z, \quad h_z = \Phi(\xi)\rho^2$$

where v_0, ρ_0, p_0 are the solution of the equations of ordinary gasdynamics, and $\Phi(\xi)$ is an arbitrary function. The conditions (1.5) and (1.7) at the shock wave also transform into the gasdynamic ones if p^* is introduced throughout.

Thus, there results a special separation of the problem into a purely hydrodynamic one together with the problem of determining the magnetic pressure.

(*Example.* Let us consider the problem of a strong explosion along a plane in an ideal conducting gas in its usual formulation [6]. For a strong explosion $p_2^* \gg p_1^*$. Neglecting p_1^* in (1.5) and (1.7) we have the conditions on the shock wave [7]

$$v_2 = \frac{2}{3}u, \quad p_2^* = \frac{2}{3}\rho_1 u^2, \quad h_{z2} = 9h_{z1}, \quad \rho_2 = 3\rho_1 \tag{2.2}$$

We take ρ_1 = const, h_{z1} = const. The solution for $v(r, t)$, $\rho(r, t)$, $p^*(r, t)$ is known [6, 8]. In addition, we have $h_z/\rho^2 = \Phi(\xi)$. Imposing the conditions at the shock we find $\Phi(\xi) = h_{z1}/\rho_1^2$. Thus

$$h_z = \left(\frac{\rho}{\rho_1}\right)^2 h_{z1}, \quad p = p^* - \left(\frac{\rho}{\rho_1}\right)^2 h_{z1}$$

The law of propagation of the shock wave is the same as in the ordinary gasdynamic case.)

3. In [9] it was shown that the solution of self-similar problems for adiabatic motion with $\nu_m = 0$ leads to the integration of two (in some cases one) ordinary equation.

A new example of a self-similar solution is given below. We consider the motion of a piston in a quiescent gas when the speed of motion of the piston is given by the law

$$U = A_1 t^n$$

The initial radius of the piston is equal to zero, and the initial values of ρ, p, h_z, h_ϕ are

$$\rho_1 = A_2 r^{-\omega}, \quad p_1 = A_3 r^{-\beta}, \quad h_{z1} = \varkappa A_3 r^{-\beta}, \quad h_{\varphi_1} = \frac{(\varkappa + 1)\beta}{2 - \beta} A_3 r^{-\beta}$$

Here A_1, A_2, A_3 are dimensional constants, and n, ω, β, χ are pure numbers.

From dimensional considerations [6] it follows that this problem with $\nu_m = 0$ is self-similar if the constants n, ω, β are connected by the relation

$$\beta = \omega - 2 + 2/(n+1)$$

Using a numerical method for integrating the system of self-similar equations [9] it is possible to solve the problem of the motion of a cylindrical piston for various values of n and ω. The results of calculations for the case of a cylindrical piston with $n = 0$, $\omega = 0$, $h_\phi = 0$, $h_{z1} = 0.025\ \rho_1 u^2$, $\gamma = 2$ and $\gamma = 5/3$ are given in the figure in the form of the dependence of the dimensionless quantities

$$v/v_2,\ \rho/\rho_2,\ p/p_2,\ h_z/h_{z2} \quad \text{on} \quad \lambda = r/r_2$$

Here the values of v/v_2 on the piston for $\gamma = 2$ and $\gamma = 5/3$ are equal to 1.451 and 1.337 respectively.

Now let $\nu_m \neq 0$; we take

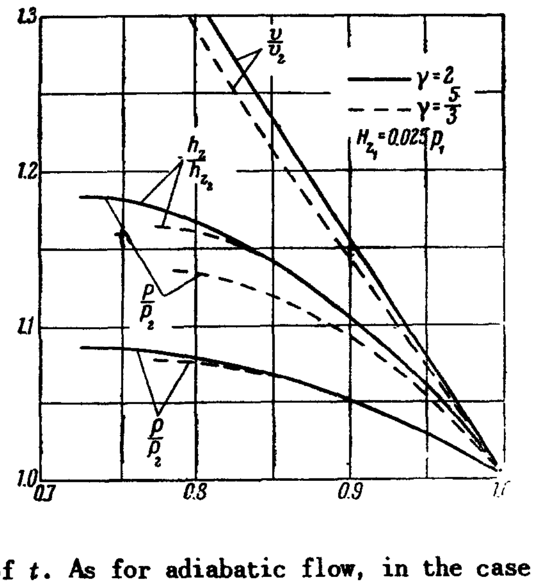
$$\nu_m = A_4 \rho^{\alpha_1} p^{\alpha_2} \quad (A_4,\ \alpha_1,\ \alpha_2 = \text{const})$$

The problem under consideration of a piston moving with constant speed is self-similar if the dimensions of A_4 depend on the dimensions of A_1 and A_2.

This is realized if the following relationship holds:

$$3\alpha_1 + \frac{\alpha_2 n + 3\alpha_2 - 1}{n+1} + (\alpha_1 + \alpha_2)(\omega - 3) + 2 = 0$$

A class of self-similar solutions exists also for the system of equations describing unsteady homothermal flows of a conducting gas. Here



$\theta(t)$ is a power of t. As for adiabatic flow, in the case under consideration with $\nu_m = 0$ there exists a "frozen" integral [9]. Therefore, the solution of all self-similar problems reduces to the integration of two ordinary equations.

4. Another class of solutions of the equations of one-dimensional magnetohydrodynamics, which we shall consider in detail, is the case of solutions for which the velocity v depends linearly on the radius.

In the case of adiabatic motion of a gas without shock waves solutions of such a type were obtained and investigated by Kulikovskii.

For arbitrary γ this solution contains one arbitrary function, and for the most interesting case $\nu = 2$ can be written in the form

$$v = \xi \frac{d\mu}{dt}, \qquad \rho = P'(\xi)(r\mu)^{-1}, \qquad P_1'(\xi) = \xi P(\xi) \qquad \left(\xi = \frac{r}{\mu}\right)$$
$$p = [b_1 P(\xi) + b_2]\mu^{-2\gamma}, \qquad h_z = [b_5 P(\xi) + b_6]\mu^{-4} \tag{4.1}$$
$$h_\varphi = \frac{1}{r^2}\{b_4 + b_3[\xi^2 P(\xi) - 2P_1(\xi)]\}$$

The function $\mu(t)$ satisfies the equation

$$(\mu')^2 = f(\mu) = \frac{b_1}{\gamma - 1}\mu^{2(1-\gamma)} - 2b_? \ln\mu + b_5\mu^{-2} + b_7$$

Here $b_1, \ldots, b_7$ are arbitrary constants, ξ is the Lagrangean coordinate, and $P(\xi)$ an arbitrary function such that $P'(\xi) > 0$. For $\nu = 2$ we have an analogous solution containing two arbitrary functions [3]

$$v = \xi \frac{d\mu}{dt}, \qquad \rho = P'(\xi)(r\mu)^{-1}$$
$$p = [B_1 P(\xi) + B_2 - \Pi(\xi)]\mu^{-4}, \qquad h_z = \Pi(\xi)\mu^{-4}$$
$$h_\varphi = \frac{1}{r^2}\{B_4 + B_3[\xi^2 P(\xi) - 2P_1(\xi)]\} \tag{4.2}$$
$$(\mu')^2 = f(\mu) = B_1\mu^{-2} - 2B_3 \ln\mu + B_5$$

where $B_1, \ldots, B_5$ are arbitrary constants, and $P(\xi)$ and $\Pi(\xi) > 0$ are arbitrary functions. The solution (4.2) was also found by Kulikovskii*.

Solutions analogous to (4.1) and (4.2) of the equations describing motion including the force of gravity were considered in [3].

Solutions of the form (4.1) and (4.2), which contain arbitrary functions, may be joined with a shock wave to the trivial solution – gas at rest – and describe the flow of gas accompanied by shock waves. In the still medium ahead of the shock wave $v_1 = 0$ and, as follows from the

* On some new exact solutions of the equations of magnetohydrodynamics. Dissertation submitted for the degree of candidate in physico-mathematical science, Moscow, 1959.

system (1.1)-(1.2), the density ρ_1 may be any function of r, and the magnetic and hydrostatic pressures are related by the equilibrium equation

$$\frac{\partial}{\partial r}(p_1 + h_1) + \frac{2h_{\varphi 1}}{r} = 0 \tag{4.3}$$

From (4.3) it follows that if $h_\phi = 0$ or $h_{\phi 1} = r^{-2}$ const, then $p_1 + h_{z1}$ = const.

In the case of a gas with $\gamma = 2$ ($h_\phi = 0$) the problem of joining the solution to still gas by means of a shock wave (which we will henceforth call the joining problem) leads to the gasdynamic one.

Now the joining problem for the gasdynamic solution of the form (4.1) with arbitrary γ was solved in [10-12]. In Shikin's paper [12] the detonation wave was also considered. As already noted, after solution of the gasdynamic problem the conditions for the discontinuity in the field (1.6) can be satisfied by the choice of the two arbitrary functions appearing in the solution. With this choice one of the functions p_1 or h_{z1} remains arbitrary. Using the results of [12], in a similar manner (for $\gamma = 2$) we obtain the solution of the problem of joining the solution (4.2) with still gas by means of a detonation wave. In view of what has been said above, we will not write out the solution of the joining problem with $\gamma = 2$ for a shock wave of arbitrary intensity. The joining problem for (4.1) can be easily solved in the limiting case of shock waves that are strong in the gasdynamic sense.

We assume that $\rho_1 u^2 \gg \gamma p_1 - 2h_1$. Then the conditions at the shock wave take the form [7]

$$\rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1, \qquad h_{z2} = \left(\frac{\gamma+1}{\gamma-1}\right)^2 h_{z1}$$
$$h_{\varphi 2} = \left(\frac{\gamma+1}{\gamma-1}\right)^2 h_{\varphi 1}, \qquad p_2 = \frac{2}{\gamma+1}\rho_1 u^2 \qquad v_2 = \frac{2}{\gamma+1}u \tag{4.4}$$

In the problem under consideration it is possible to use the arbitrary function $P(\xi)$ appearing in the solution (4.1) and also the arbitrarily assigned $\rho_1(r)$, $h_{z1}(r)$, $h_{\phi 1}(r)$ and the speed of the shock wave $u(r_2)$. With the choice of these five functions one can satisfy all the conditions at the shock wave (4.4).

Satisfying the conditions at the shock, we find that the arbitrary function $P(\xi)$ should have the form

$$P(\xi) = \left[b_9 + \frac{\alpha b_2}{b_1}\int \exp\left(-\alpha\int\frac{d\xi}{f(\xi)\,\xi^5}\right)\frac{d\xi}{f(\xi)\,\xi^5}\right]\exp\left(\alpha\int\frac{d\xi}{f(\xi)\,\xi^5}\right) \tag{4.5}$$

where

$$f(\xi) = \frac{\alpha}{2}\xi^{-4} - \frac{4}{\gamma-1} b_3 \ln\frac{\xi}{b_8} + b_5\left(\frac{\xi}{b_8}\right)^{-\frac{4}{\gamma-1}} + b_7, \quad \alpha = \frac{2b_1 b_8^{4}}{\gamma-1} \quad (b_9 = \mathrm{const})$$

The law of motion of the shock wave is given by the relation

$$r_2 = b_8 \mu^{1/2(\gamma+1)} \qquad (b_8 = \mathrm{const})$$

Thus

$$\rho_1(r) = \frac{\gamma-1}{\gamma+1}\frac{1}{r}\left(\frac{r}{b_8}\right)^{-\frac{2}{\gamma+1}} \frac{dP}{dx}, \quad h_{z1}(r) = \left(\frac{\gamma-1}{\gamma+1}\right)^2 [b_5 P(\xi) + b_3]\left(\frac{r}{b_8}\right)^{-\frac{\varkappa}{\gamma+1}}$$

$$h_{\varphi 1}(r) = \left(\frac{\gamma-1}{\gamma+1}\right)^2 \left\{\frac{b_4}{r^2} + b_3\left[\left(\frac{r}{b_8}\right)^{-\frac{4}{\gamma+1}} P(x) - 2r^{-2}P_1(x)\right]\right\}, \qquad x = b_8^{\frac{2}{\gamma+1}} r^{\frac{\gamma-1}{-}}$$

where $P(x)$ is given by the relation (1.5).

In order that the equilibrium equation may be satisfied, the initial pressure p_1 must depend on r in the following way:

$$p_1(r) = b_{10} - h_{z1}(r) - h_{\varphi 1}(r) - 2\int \frac{h_{\varphi 1}(r)}{r} dr \quad (b_{10} = \mathrm{const})$$

The set of constants appearing here can be used to satisfy the other conditions exactly or approximately in concrete cases.

The joining problem can be solved also for arbitrary γ and shock waves of arbitrary intensity with variable h_{z1}, p_1, ρ_1 and $h_\phi = 0$. Its solution can be found, for example, by the method of joining of [12].

(*Example.* Let us consider the example of the application of the solution (4.2) to the problem of the motion of gas being compressed by a piston moving with speed

$$U = R\,\frac{\mu'}{\mu} = \pm\frac{R}{\mu}\sqrt{f(\mu)}, \qquad (R(t) \text{ is the radius of the piston})$$

The minus sign corresponds to motion toward the center (axis of symmetry), and the plus sign to motion away from the center (axis). For simplicity we will take the shock wave to be strong. Let the initial values of ρ_1, p_1, h_{z1} be

$$\rho_1 = A_0 r^{2/3}, \qquad p_1 = L - h_{z1}, \qquad h_{z1} = L_1 r^{-8/3} + L_2 \qquad (A_0,\ L,\ L_1,\ L_2 = \mathrm{const})$$

Using (4.2) and (2.2), we find that the flow of gas behind the shock wave is determined by the formulas

$$v = \pm\frac{B_1^{1/2} r}{B_7 \pm 2B_1^{1/2} t}, \qquad \rho = 3A_0 B_6^{8}\xi^{6}\mu^{-2}, \qquad p = 3B_6^{8}\left(\frac{1}{8} A_0 B_1 - 3B_6^{4} L_2\right)\xi^{8}\mu^{-4}$$

$$r_2=B_6^{-3/2}\mu^{3/2}, \qquad h_z = 9B_6^4 (L_1 + B_6^8 L_2 \xi^8)\, \mu^{-4}, \qquad \mu^2 = B_7 \pm 2B_1^{1/2} t \qquad (4.6)$$

Here B_7 is an arbitrary constant. Among the constants A_0, B_1, B_6, L_2 there is the relationship

$$8L_2 + A_0 B_1 B_6^{-4} = 0$$

For the azimuthal field creating the magnetic field h_2 we have

$$j_\varphi = \frac{9A_0B_1B_6^8}{\sqrt{8\pi h_z}}\, cr^7\mu^{-12} \quad (c \text{ is the speed of light})$$

This follows from the formula

$$\frac{4\pi}{c}\, j_\varphi = -\frac{\partial H_z}{\partial r}$$

The law of motion of the piston is given by the relation

$$R = R_0 \left(1 \pm 2\, \frac{B_1^{1/2}}{B_7}\, t\right)^{1/2}$$

where R_0 is the radius of the piston at the initial instant $t = 0$.

For the converging piston we have the initial value of the velocity

$$U_0 = -\frac{B_1^{1/2}}{B_7}\, R_0$$

Such a family of motions can be created, for example, by the application to a gas of an external magnetic field. If there is a vacuum outside the piston, then the total pressure on the piston should be compensated by the application of an external magnetic field h_e such that the relation $p^*(R, t) = H_e^2/8\pi = h_e$ is satisfied. Naturally the solution (4.6) describes the flow for those values of r where $p_1 > 0$, $h_{z1} > 0$.)

5. In the case of homothermal flows the system of equations (1.1), (1.4) with $\nu_m = 0$ also has exact solutions, where $v(r, t)$ has the same form as in (4.1). These solutions contain either arbitrary functions of ξ or arbitrary functions of the time [1].

For homothermal flows with $\nu_m \neq 0$ one may take $\nu_m = \nu_m(t)$. Then by using the choice of the arbitrary function $P(\xi)$ appearing in the solution of the equations for an ideal medium [1] it is possible to find a particular solution of the full system (1.1), (1.4) when h_z or h_ϕ is absent. This solution will contain a certain number of arbitrary constants.

For homothermal flows with linear dependence of velocity on radius (for $\nu_m = 0$) one can also consider the problem of joining the exact solution with still gas through a shock wave. The case when the solution contains an arbitrary function of time was considered previously [1].

We write the particular solution of the system (1.1), (1.4) in the form [1]

$$v = \frac{r}{\mu(t)}\frac{d\mu}{dt}, \quad \rho = \frac{P'(\xi)}{r\mu}, \quad \theta(\mu) = \delta_1\mu^{-2}, \qquad h_\varphi = \frac{\delta_1}{r^2} \tag{5.1}$$

$$h_z = \left[\delta_2 - \delta_3 P(\xi) - \delta_1 \frac{P'(\xi)}{\xi}\right]\mu^{-4}, \quad p = \theta(\mu)\rho, \qquad f(\mu) = (\mu')^2 = \delta_5 - \delta_3\mu^{-2}$$

where $\delta_1, \ldots, \delta_5$ are arbitrary constants. We will take $\delta_4 = 0$. Using the relation

$$u - v_2 = u\left(1 - \frac{r_2}{\mu}\frac{d\mu}{dr_2}\right), \qquad \frac{d\xi_2}{d\mu} = \frac{1}{\mu}\left[\frac{dr_2}{d\mu} - \frac{r_2}{\mu}\right]$$

from the solution (5.1) and relations (1.5), (1.6) we have

$$P(r_2) = \int \rho_1(r_2)\, r_2 dr_2 \tag{5.2}$$

$$\int \rho_1(r_2)\, r_2 dr_2 = \frac{\mu}{\sigma(\mu)}\left\{k_1 + \frac{\delta_2}{\delta_3}\frac{\sigma(\mu)}{\mu} - \frac{p_1^*}{2\delta_5}\mu\sigma(\mu) - \frac{p_1^*}{2\delta_5}\left(\frac{\delta_3}{\delta_5}\right)\ln[\sigma(\mu) + \mu]\right\}$$

$$\sigma(\mu) = \left(\mu^2 - \frac{\delta_3}{\delta_5}\right)^{1/2} \qquad (k_1 = \mathrm{const}) \tag{5.3}$$

The relation (5.3) gives the law of motion of the shock wave.

Satisfying conditions (1.6) we find for the discontinuity in magnetic field

$$h_{z_1}(r) = h_{z_1}(r_2) = \mu^{-4}\left[(\delta_2 - \delta_3 P(r_2))\left(1 - \frac{r_2}{\mu}\frac{d\mu}{dr_2}\right) - \delta_1\mu^2\rho_1(r_2)\right]\left(1 - \frac{r_2}{\mu}\frac{d\mu}{dr_2}\right) \tag{5.4}$$

where $\mu(r_2)$ is a known function from (5.3).

The dependence $P(r_2)$ is found from (5.2) if $\rho_1(r)$ is considered a given function. For the full solution of the problem, however, it is necessary to know $P(\xi)$.

Using (5.3) and the form of the function $P(r_2)$, we can find the dependence $P(\xi)$. It is given by the relations

$$P(\xi) = P[r_2(\xi)]$$

$$\begin{aligned} \frac{\xi\sigma(r_2, \xi)}{r_2}\int \rho_1(r_2)\, r_2 dr_2 = k_1 + \frac{\delta_2}{\delta_3}\frac{\sigma(r_2, \xi)}{r_2}\xi - \\ - \frac{p_1^* r_2}{2\delta_5\xi}\sigma(r_2, \xi) - \frac{p_1^*}{2\delta_5}\left(\frac{\delta_3}{\delta_5}\right)\ln\left[\sigma(r_2, \xi) + \frac{r_2}{\xi}\right] \end{aligned} \tag{5.5}$$

$$\sigma(r_2, \xi) = \left(r_2^2\xi^{-2} - \frac{\delta_3}{\delta_5}\right)^{1/2}$$

We consider further one special case. Let $p_1^* = 0$ (a strong wave) and $\rho_1 = \text{const}$. In this case for the function $P(\xi)$ and the law of motion of the shock wave we have the relations

$$2[P(\xi)-B] = \rho_1\xi^2 \frac{\delta_3}{\delta_5}\left[P(\xi) - \frac{\delta_2}{\delta_3}\right]^2\left[\left(P(\xi)-\frac{\delta_2}{\delta_3}\right)^2 - k_1^2\right]^{-1}$$

$$\mu^2 = \frac{\delta_3}{\delta_5}\left(\frac{\rho_1 r_2^2}{2} + B - \frac{\delta_2}{\delta_3}\right)^2\left[\left(\frac{\rho_1 r_2^2}{2} + B - \frac{\delta_2}{\delta_3}\right)^2 - k_1^2\right]^{-1}$$

The initial magnetic field is given by the formula

$$h_{z1} = \left(\frac{\delta_5}{\delta_3}\right)^2\left(\frac{\chi - k_1^2}{\chi}\right)^2\left[\delta_2 - \delta_3\left(\frac{\rho_1 r^2}{2} + B\right)\right]\left[1 + \frac{k_1^2\rho_1 r^2}{\chi^{1/2}(\chi - k_1^2)}\right]^2 -$$

$$- \frac{\delta_5\delta_1}{\delta_3}\rho_1\frac{\chi - k_1^2}{\chi}\left[1 + \frac{k_1^2\rho_1 r^2}{\chi^{1/2}(\chi - k_1^2)}\right]$$

$$\chi = \chi(r) = \left(\frac{\rho_1 r^2}{2} + B - \frac{\delta_2}{\delta_3}\right)^2 \qquad (B = \text{const})$$

From (5.1) we find the dependence $\mu(t)$:

$$\mu(t) = \sqrt{\delta_5(t-t_0)^2 + \frac{\delta_3}{\delta_5}} \qquad (t_0 = \text{const}) \tag{5.6}$$

The solution (5.1)-(5.6) can be used for the investigation of concrete problems and, in particular, can have application to the problem of compression of gas by a converging or diverging piston, where the law of motion of the piston is given by the dependence

$$R = A\sqrt{\delta_5(t-t_0)^2 + \frac{\delta_3}{\delta_5}} \qquad (A = \text{const})$$

Considering the remarks made in Section 4, one can say that the problem of compression of gas by a converging piston is equivalent to the problem of an impulsive gaseous discharge with passage of current through a gaseous cylinder in the axial direction. The magnetic field of the current flowing on the surface of the cylinder will play the role of a piston compressing the gas.

6. (*Example.* Let us consider the exact solution of the problem of an impulsive gaseous discharge. Let there be at the initial moment $t = 0$ a cylindrical column of gas heated to a temperature at which the conductivity of the gas may be considered infinite. In the gas there is a "frozen in" magnetic field with vector intensity **H** parallel to the axis of the cylinder. The initial density of the gas is constant and equal to ρ_1, and the total pressure in the gas $p_1^* = \text{const}$.

The initial intensity of the magnetic field is constant with radius, its dependence on the coordinate r being given by the formula

$$h_{z_1} = \left[\beta_2^2\left(1 - \frac{\beta_3^2\rho_1 r^2\varphi(r)}{\beta_2^2 - p_1^*\varphi^2(r)}\right) - \beta_4^2\rho_1\varphi(r)\right]\left(1 - \frac{\beta_3^2\rho_1 r^2\varphi(r)}{\beta_2^2 - p_1^*\varphi^2(r)}\right)[\varphi(r)]^{-2} \qquad (6.1)$$

where $\phi(r)$ is determined from the equation

$$p_1^*\varphi^2 - 2\beta_3^2(\beta_5 - \tfrac{1}{2}\rho_1 r^2)\varphi + \beta_2^2 = 0 \qquad (\beta_1, \ldots, \beta_5 = \text{const}, \ \beta_2 > 0, \ \beta_3 > 0)$$

The choice of the dependence of h_{z1} on r in the form (6.1) is based on the subsequent use of the solution (5.1)-(5.6) with $\delta_3 = 0$. Formula (6.1) is essentially a consequence of the relation (5.4) with the arbitrary constants re-designated.

Let the initial radius of the gas cylinder be R_0. At the moment $t = 0$ a current begins to pass along the column in the axial direction, varying with time so that the total current is given by the formula

$$I = \frac{\sqrt{2\pi}\,\beta_1\beta_2}{\beta_0 - \beta_3 t}c \qquad (\beta_0, \ \beta_1 = \text{const} > 0)$$

For $t > 0$ as a consequence of the pinch-effect there begins a contraction of the plasma sheath outside of which $p = 0$. A shock wave will propagate toward the center of the gas. It is required to determine the motion of the gas between the shock wave and the outer radius of the gas column.

In a formulation different from that stated above (in the sense of the given initial conditions) the problem of the compression of a gaseous cylinder by a current was considered in a series of works (see, for example, [13-14]).

We will consider the problem in the approximation of magnetohydrodynamics and suppose that the gradient of temperature in the region behind the shock wave is equal to zero. The quantities p, ρ, h_z and v behind the wave front are related by the equations of magnetohydrodynamics (1.1), (1.4); on the shock wave front itself the laws (1.5), (1.6), which are the boundary conditions for the unknown functions, must be satisfied.

On the outer surface of the cylinder there must also be satisfied the kinematic condition $v(R, t) = dR/dt$, where $R(t)$ gives the dependence of the radius of the cylinder on time. The solution satisfying the system (1.1), (1.4) and the initial and boundary conditions enumerated above has the form

$$v = -\beta_3\xi \qquad \left(\xi = \frac{r}{\beta_0 - \beta_3 t}\right), \qquad p = \theta(t)\rho, \qquad \theta(t) = \frac{\beta_4^2}{(\beta_0 - \beta_3 t)^2}$$

$$\rho = \frac{F(\xi)}{(\beta_0 - \beta_3 t)^2}, \qquad h_z = (\beta_2^2 - \beta_4^2 F(\xi))(\beta_0 - \beta_3 t)^{-4} \qquad (6.2)$$

$$F(\xi) = \frac{\beta_2^2(\rho_1\xi^2 + p_1^*\beta_3^{-2}) - p_1^*[2\beta_5 y - (\beta_2\beta_3^{-1})^2]}{2\beta_3^2(\beta_5 - p_1^*\beta_3^{-2} - \rho_1\xi^2 y)(\xi^2 + p_1^*\beta_3^{-2}\rho_1^{-1})}$$

where $y(\xi)$ is found from the equation

$$y^2(\rho_1\xi^2 + \beta_3^{-2}p_1^*) - 2\beta_5 y + (\beta_2\beta_3^{-1})^2 = 0$$

For the radius R of the cylinder and the radius r_2 of the shock wave we have

$$R(t) = \beta_1(\beta_0 - \beta_3 t), \qquad R_0 = \beta_1\beta_0$$

$$r_2(t) = 2\beta_5\rho_1^{-1} - \beta_3^{-2}\rho_1^{-1}[\beta_2^2 + p_1^*(\beta_0 - \beta_3 t)^4](\beta_0 - \beta_3 t)^{-2}$$

From (6.2) it is clear that the temperature in the region of motion of the gas grows proportional to $(\beta_0 - \beta_3 t)^{-2}$. If we suppose that $p_2^* \gg p_1^*$, then the quantity p_1^* in (1.5) may be neglected. The formulas giving the solution are, thus simplified.

The solution (6.2) was obtained with the choice, described in Section 5, of (5.1) with $\delta_3 = 0$. From (6.2) it follows that the total pressure on the outer boundary of the cylinder p_1^* is equal to the magnetic pressure outside the cylinder

$$h_\varphi = \frac{I^2}{2\pi c^2 R^2}$$

arising from the passage of the current I over the surface of the cylinder.

Since in the region of flow we took $\partial T/\partial r = 0$, the differential equation (1.3) expressing the law of conservation of energy of a particle is not satisfied. However, it is possible to demand that the integral law of conservation of energy be satisfied. If the initial energy of the undisturbed gas is neglected, which corresponds to the case of a strong shock wave, then the balance of energy for the motion of the gas is given by the relation

$$2\pi \int_{r_2}^{R} \left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + h_z\right) r\,dr + Q(t) = 2\pi \int_0^t p^*(R, t) R \frac{dR}{dt}\,dt \qquad (6.3)$$

In (6.3) $Q(t)$ signifies that part of the energy which is added to the gas, for example as a result of chemical or nuclear transformations, and removed from the outer surface as a result of radiation. On the right-hand side of (6.3) is the work of the total pressure force on the boundary of the cylinder at time t. Let $\gamma = 2$ for simplicity. Then using (6.2) and (6.3) we find for the quantity $Q(t)$

$$\frac{Q}{2\pi\beta_2^2} = \left(\frac{\beta_1}{\beta_0}\right)^2 + \frac{\beta_5}{2\rho_1 y_0^2} - \frac{\beta_2^2}{6\rho_1\beta_3^2 y_0^3} - \frac{2\beta_1^2}{\beta_4^2}\theta - \frac{\beta_5}{\rho_1\beta_4^4}\left(4\frac{\beta_5}{\rho_1} - \frac{1}{2}\right)\theta^2 +$$
$$+ \frac{\beta_2^2}{\beta_3^2\beta_4^6\rho_1}\left(\frac{1}{6} - 4\frac{\beta_5}{\rho_1}\right)\theta^3 + \frac{\beta_2^4}{\beta_3^4\beta_4^8\rho_1^2}\theta^4$$

We note that the solution of the equations of magnetohydrodynamics with $\gamma = 2$ with application to other problems was considered also by other authors (see, for example, [15]).)

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